Journal of Engineering Mathematics, Vol. 3, No. 4. October 1969
Wolters-Noordhoff Publishing – Groningen
Printed in the Netherlands

Optimal Spin Recovery

G. R. WALSH

Department of Mathematics, University of York, Heslington, York, England

(Received April 4, 1969)

SUMMARY

When an aircraft has developed a spin, the pilot is usually concerned not only with recovering from the spin but doing so with minimum loss of height. This paper is mainly concerned with the mathematical formulation of this problem, including the simplifications that can reasonably be made.

The Pontryagin minimum principle is used to obtain necessary conditions for a solution to the problem. The amount of computation required may be prohibitive; on the other hand, if certain simplifying assumptions may be regarded as valid, it is indicated how a solution may be obtained.

The theory is restricted to the case in which three controls are actively used: aileron, elevator and rudder. However, further controls may easily be included, with only a slight increase in complexity.

1. Introduction

The spinning of aircraft and the control movements necessary to recover from the spin have been the subject of many detailed studies [1], [2], [3], [4]. It is also possible to obtain good approximations to spinning motions with relatively little mathematical detail [5]. The main emphasis in such studies is usually centred on the question of whether "normal recovery action" is satisfactory, the objective being to recover from the spin as quickly and smoothly as possible. The definition of recovery cannot be exact. An aircraft may be considered to have recovered from a spin when the incidence of the centre section of the wing is below the stall, though this condition may be accompanied by large residual angular velocities in roll and yaw. Another definition may include some restrictions on these angular velocities.

In this paper, emphasis is placed upon the loss of height during the attempted recovery from the spin. This parameter is certainly important and may be critical. The definition of recovery is left open to the extent that, in the present formulation of the problem, different definitions merely imply different sets of terminal conditions.

2. Equations of Motion

A complete list of notation is given at the end of the paper.

The general equations of motion of an aircraft are: [6]

$$\dot{u} + qw - rv - g_x = X/m\,, \tag{1}$$

$$\dot{v} + ru - pw - g_y = Y/m\,, \tag{2}$$

$$\dot{w} + pv - qu - g_z = Z/m\,, \tag{3}$$

$$\dot{p} + d_x(q^2 - r^2) + e_x(\dot{r} + pq) + f_x(\dot{q} - rp) + b_x qr + c_{zx} q - c_{yx} r = L/I_x\,, \tag{4}$$

$$\dot{q} + e_y(r^2 - p^2) + f_y(\dot{p} + qr) + d_y(\dot{r} - pq) + b_y rp + c_{xy} r - c_{zy} p = M/I_y\,, \tag{5}$$

$$\dot{r} + f_z(p^2 - q^2) + d_z(\dot{q} + rp) + e_z(\dot{p} - qr) + b_z pq + c_{yz} p - c_{xz} q = N/I_z\,. \tag{6}$$

The kinematic relationships are:

$$p = \dot{\phi} - \dot{\psi} \sin\theta\,, \tag{7}$$

$$q = \dot{\theta} \cos\phi + \dot{\psi} \sin\phi \cos\theta\,, \tag{8}$$

$$r = -\dot{\theta} \sin\phi + \dot{\psi} \cos\phi \cos\theta\,. \tag{9}$$

The following assumptions are made throughout the subsequent work:
The engine momentum parameters c_{zx}, etc. are all zero.
The product of inertia parameters d_x, e_x, f_x, etc. are all zero.

$$X = L \sin \alpha - D \cos \alpha + T \cos \theta_T , \tag{10}$$
$$Y = Y_e + Y' , \tag{11}$$
$$Z = -L \cos \alpha - D \sin \alpha - T \sin \theta_T , \tag{12}$$
$$L = L_e + L' , \tag{13}$$
$$M = M_e + M' , \tag{14}$$
$$N = N_e + N' , \tag{15}$$

where

$$T = T_e + T' \tag{16}$$

and

$$\sin \alpha = \frac{w}{(u^2+w^2)^{\frac{1}{2}}} , \quad \cos \alpha = \frac{u}{(u^2+w^2)^{\frac{1}{2}}} . \tag{17}$$

The datum values, denoted by suffix e, refer to a steady spin defined by

$$u = u_e, \quad v = v_e, \quad p = p_e, \quad q = q_e, \quad r = r_e ,$$
$$\phi = \phi_e, \quad \theta = \theta_e, \quad \dot{\psi} = \dot{\psi}_e, \quad \alpha = \alpha_e, \quad T = T_e .$$

Note that the relatively small terms $X_q q$, $X_{\dot{w}} \dot{w}$, $X_\eta \eta$, $Z_q q$, $Z_{\dot{w}} \dot{w}$ and $Z_\eta \eta$ have been neglected. Substituting the datum values into equations (1)–(6) and (10)–(17) we obtain

$$q_e w_e - r_e v_e + g \sin \theta_e = (L_e \sin \alpha_e - D_e \cos \alpha_e + T_e \cos \theta_T)/m , \tag{18}$$
$$r_e u_e - p_e w_e - g \sin \phi_e \cos \theta_e = Y_e/m , \tag{19}$$
$$p_e v_e - q_e u_e - g \cos \phi_e \cos \theta_e = (-L_e \cos \alpha_e - D_e \sin \alpha_e - T_e \sin \theta_T)/m , \tag{20}$$
$$b_x q_e r_e = L_e/I_x , \tag{21}$$
$$b_y r_e p_e = M_e/I_y , \tag{22}$$
$$b_z p_e q_e = N_e/I_z . \tag{23}$$

Equations (21)–(23) may be regarded as equations to determine the steady applied moments L_e, M_e, N_e in terms of the angular velocity components in the steady spin p_e, q_e, r_e.
From equations (7), (8), (9), we have

$$p_e = -\dot{\psi}_e \sin \theta_e, \quad q_e = \dot{\psi}_e \sin \phi_e \cos \theta_e, \quad r_e = \dot{\psi}_e \cos \phi_e \cos \theta_e . \tag{24}$$

Substituting from equations (24) into equations (18)–(20), and using equation (17), we obtain

$$\frac{w_e L_e - u_e D_e}{m(u_e^2 + w_e^2)^{\frac{1}{2}}} = q_e w_e - r_e v_e - \frac{g p_e}{\dot{\psi}_e} - \frac{T_e}{m} \cos \theta_T , \tag{25}$$

$$\frac{Y_e}{m} = r_e u_e - p_e w_e - \frac{g q_e}{\dot{\psi}_e} , \tag{26}$$

$$\frac{-u_e L_e - w_e D_e}{m(u_e^2 + w_e^2)^{\frac{1}{2}}} = p_e v_e - q_e u_e - \frac{g r_e}{\dot{\psi}_e} + \frac{T_e}{m} \sin \theta_T . \tag{27}$$

Equations (25)–(27) determine the steady lift, drag and sideforce L_e, D_e, Y_e in terms of u_e, v_e, w_e, p_e, q_e, r_e, $\dot{\psi}_e$, T_e.
Next, we require expressions for L, D and the primed quantities in equations (10)–(15). In equations (10) and (12), assume for simplicity that

$$L = \tfrac{1}{2}\rho S V^2 C_L(\alpha), \quad D = \tfrac{1}{2}\rho S V^2 C_D(\alpha) , \tag{28}$$

where

$$V^2 = u^2 + w^2 . \tag{29}$$

This assumption has been shown to be valid for large-disturbance manoeuvres [7]. It is assumed that C_L and C_D are known functions of α, and that ρ is a known function of altitude, i.e. $\rho = \rho(h)$, where

$$h = h_e - h_L(t) , \tag{30}$$

and $h_L(t)$ is the height lost in time t, given by

$$h_L(t) = \int_0^t [-u(\tau) \sin \theta(\tau) + v(\tau) \sin \phi(\tau) \cos \theta(\tau) + w(\tau) \cos \phi(\tau) \cos \theta(\tau)] d\tau . \tag{31}$$

According to Perkins and Hage [8],

$$\rho = \rho_0 \left(1 - \frac{\delta h}{T_0}\right)^{(1/\delta R) - 1} , \tag{32}$$

within the troposphere, where

$$\begin{aligned} \rho_0 &= 0.002378 \text{ slug/ft}^3., \\ \delta &= 0.003566\,^\circ\text{F/ft.}, \\ T_0 &= 518.4\,^\circ\text{F}, \\ R &= 53.36 \text{ ft./}^\circ\text{F}. \end{aligned}$$

For the general motion, let

$$\left.\begin{aligned} &u = u_e + u', \quad v = v_e + v', \quad w = w_e + w' , \\ &p = p_e + p', \quad q = q_e + q', \quad r = r_e + r', \quad \dot\psi = \dot\psi_e + \dot\psi' , \end{aligned}\right\} \tag{33}$$

and assume

$$Y' = Y_v v' , \tag{34}$$

$$L' = L_v v' + L_p p' + L_\xi \xi' , \tag{35}$$

$$M = M_e + M' = \tfrac{1}{2}\rho S l V^2 C_m(\alpha) + M_q (q_e + q') + M_\eta(\eta_e + \eta') , \tag{36}$$

$$N' = N_v v' + N_r r' + N_\zeta \zeta' , \tag{37}$$

where ξ', η', ζ' are the control angles measured from the datum values, i.e.

$$\xi = \xi_e + \xi', \quad \eta = \eta_e + \eta', \quad \zeta = \zeta_e + \zeta' . \tag{38}$$

In the steady spin we must have

$$Y_e = Y_v v_e , \tag{39}$$

$$L_e = L_v v_e + L_p p_e + L_\xi \xi_e , \tag{40}$$

$$M_e = \tfrac{1}{2}\rho_e S l V_e^2 C_m(\alpha_e) + M_q q_e + M_\eta \eta_e , \tag{41}$$

$$N_e = N_v v_e + N_r r_e + N_\zeta \zeta_e , \tag{42}$$

$$L_e = \tfrac{1}{2}\rho_e S V_e^2 C_L(\alpha_e) , \tag{43}$$

$$D_e = \tfrac{1}{2}\rho_e S V_e^2 C_D(\alpha_e) . \tag{44}$$

In equation (36), $C_m(\alpha)$ is the pitching moment coefficient for $\eta = 0$ and $q = 0$.

The datum conditions are given by equations (21)–(23), (25)–(27) and (39)–(44). Eliminating the forces Y_e, L_e, D_e and the moments L_e, M_e, N_e between these equations, we obtain

$$\frac{\tfrac{1}{2}\rho_e S V_e^2}{m} [C_L(\alpha_e) \sin \alpha_e - C_D(\alpha_e) \cos \alpha_e] = q_e w_e - r_e v_e - \frac{g p_e}{\dot\psi_e} - \frac{T_e}{m} \cos \theta_T , \tag{45}$$

$$\frac{Y_v}{m} v_e = r_e u_e - p_e w_e - \frac{g q_e}{\dot\psi_e} , \tag{46}$$

$$\frac{\frac{1}{2}\rho_e S V_e^2}{m}\left[-C_L(\alpha_e)\cos\alpha_e - C_D(\alpha_e)\sin\alpha_e\right] = p_e v_e - q_e u_e - \frac{g r_e}{\dot{\psi}_e} + \frac{T_e}{m}\sin\theta_T\,, \tag{47}$$

$$\frac{1}{I_x}\left[L_v v_e + L_p p_e + L_\xi \xi_e\right] = b_x q_e r_e\,, \tag{48}$$

$$\frac{1}{I_y}\left[\tfrac{1}{2}\rho_e S l V_e^2 C_m(\alpha_e) + M_q q_e + M_\eta \eta_e\right] = b_y r_e p_e\,, \tag{49}$$

$$\frac{1}{I_z}\left[N_v v_e + N_r r_e + N_\zeta \zeta_e\right] = b_z p_e q_e\,. \tag{50}$$

In equations (45)–(50), we have the relations

$$V_e^2 = u_e^2 + w_e^2\,, \tag{51}$$

$$\sin\alpha_e = \frac{w_e}{(u_e^2+w_e^2)^{\frac{1}{2}}}\,, \qquad \cos\alpha_e = \frac{u_e}{(u_e^2+w_e^2)^{\frac{1}{2}}}\,. \tag{52}$$

From equations (24), we find

$$\dot{\psi}_e = \operatorname{sgn}(r_e)(p_e^2+q_e^2+r_e^2)^{\frac{1}{2}}\,, \tag{53}$$

since both ϕ_e and θ_e are assumed to be restricted to the range $(-\pi/2, \pi/2)$.

In a steady spin, the dynamical equations (45)–(50) must be satisfied, while equations (24) provide the necessary kinematic relationships. It may be said [1] that

(i) the balance of vertical forces determines the rate of descent,
(ii) the balance of pitching moments determines the rate of rotation,
(iii) the balance of rolling moments determines the sideslip,
(iv) the balance of yawing moments determines the possible combinations of rate of rotation, sideslip and incidence for a steady spin. If the yawing moments cannot be balanced then no steady spin is possible.

In the sequel, we shall assume that the conditions for a steady spin are satisfied.

Returning to the equations of motion (1)–(6) and using equations (10)–(16), (28) and (33)–(42), we obtain

$$\dot{u} = -qw + rv - g\sin\theta + \frac{1}{m}\left[\tfrac{1}{2}\rho S V^2(C_L\sin\alpha - C_D\cos\alpha) + T\cos\theta_T\right], \tag{54}$$

$$\dot{v} = -ru + pw + g\sin\phi\cos\theta + \frac{Y_v v}{m}, \tag{55}$$

$$\dot{w} = -pv + qu + g\cos\phi\cos\theta - \frac{1}{m}\left[\tfrac{1}{2}\rho S V^2(C_L\cos\alpha + C_D\sin\alpha) + T\sin\theta_T\right], \tag{56}$$

$$\dot{p} = -b_x qr + \frac{1}{I_x}(L_v v + L_p p + L_\xi \xi)\,, \tag{57}$$

$$\dot{q} = -b_y rp + \frac{1}{I_y}(\tfrac{1}{2}\rho S l V^2 C_m + M_q q + M_\eta \eta)\,, \tag{58}$$

$$\dot{r} = -b_z pq + \frac{1}{I_z}(N_v v + N_r r + N_\zeta \zeta)\,. \tag{59}$$

We also need the two kinematic equations obtained by eliminating $\dot{\psi}$ between equations (7), (8) and (9). These are

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta\,, \tag{60}$$

$$\dot{\theta} = q\cos\phi - r\sin\phi\,. \tag{61}$$

Equations (54)–(61) are essentially eight equations for the eight unknowns $u, v, w, p, q, r, \phi, \theta$. It is assumed that the function $C_m(\alpha)$ is known, as well as the functions $C_L(\alpha)$ and $C_D(\alpha)$. However, in the case when air density is assumed to vary with height, we also need the equation

$$\dot{h} = u \sin\theta - v \sin\phi \cos\theta - w \cos\phi \cos\theta \,, \tag{62}$$

which comes from equations (30) and (31). Then equations (54)–(62), together with equations (29) and (32), form the complete set of state equations. The controls are ξ, η, ζ, T.

3. Optimisation Problems

It is now possible to formulate a series of optimisation problems by assuming various initial and final conditions for the state variables. In the most general case the problem is as follows.

Given the state equations (54)–(62), with initial conditions

$$u = u_e,\ v = v_e,\ \ldots\ \theta = \theta_e,\ h = h_e \,,$$

find the controls $\xi(t), \eta(t), \zeta(t), T(t)$ which transfer the initial state to the final state with minimum loss of height. The constraints on the controls are assumed to be

$$|\xi| \leqq \xi_M, \quad \eta_m \leqq \eta \leqq \eta_M, \quad |\zeta| \leqq \zeta_M, \quad 0 \leqq T \leqq T_M \,, \tag{63}$$

where

$$\eta_m < 0, \quad \eta_M > 0 \,.$$

It is assumed for simplicity that there are no constraints on the state variables; the modifications to the Pontryagin principle when such constraints are included are described in reference [9]. The terminal time t_1 is free. Thus we have to minimise $h_L(t_1)$, given by equation (31).

Several sets of final conditions on the state variables may be assumed; these will be considered later (Section 4) since they only affect the terminal transversality conditions.

The Hamiltonian [10] is

$$\begin{aligned} H = {} & (-u \sin\theta + v \sin\phi \cos\theta + w \cos\phi \cos\theta)(1 - p_h) \\ & + \left[-qw + rv - g \sin\theta + \frac{1}{m}\left\{\tfrac{1}{2}\rho S V^2 (C_L \sin\alpha - C_D \cos\alpha) + T \cos\theta_T\right\}\right] p_u \\ & + \left[-ru + pw + g \sin\phi \cos\theta + \frac{Y_v v}{m}\right] p_v \\ & + \left[-pv + qu + g \cos\phi \cos\theta - \frac{1}{m}\left\{\tfrac{1}{2}\rho S V^2 (C_L \cos\alpha + C_D \sin\alpha) + T \sin\theta_T\right\}\right] p_w \\ & + \left[-b_x qr + \frac{1}{I_x}(L_v v + L_p p + L_\xi \xi)\right] p_p \\ & + \left[-b_y rp + \frac{1}{I_y}(\tfrac{1}{2}\rho S l V^2 C_m + M_q q + M_\eta \eta)\right] p_q \\ & + \left[-b_z pq + \frac{1}{I_z}(N_v v + N_r r + N_\zeta \zeta)\right] p_r \\ & + [p + q \sin\phi \tan\theta + r \cos\phi \tan\theta] p_\phi \\ & + [q \cos\phi - r \sin\phi] p_\theta \,. \end{aligned} \tag{64}$$

The coefficients of ξ, η, ζ, T, respectively, in H are:

$$\frac{L_\xi p_p}{I_x}, \quad \frac{M_\eta p_q}{I_y}, \quad \frac{N_\zeta p_r}{I_z}, \quad \frac{1}{m}(p_u \cos\theta_T - p_w \sin\theta_T) \,.$$

Since L_ξ, M_η and N_ζ are all negative, the Pontryagin minimum principle [10] states that the optimal controls are

$$\xi = \text{sgn}\,(p_p)\xi_M\,, \tag{65}$$

$$\eta = \eta_M \text{ if } \text{sgn}\,(p_q) > 0\,,$$
$$= \eta_m \text{ if } \text{sgn}\,(p_q) < 0\,,$$

i.e. $$\eta = \tfrac{1}{2}[(\eta_M+\eta_m)+\text{sgn}\,(p_q)(\eta_M-\eta_m)]\,, \tag{66}$$

$$\zeta = \text{sgn}\,(p_r)\zeta_M\,, \tag{67}$$

$$T = 0 \quad \text{if } \text{sgn}\,(p_u \cos\theta_T - p_w \sin\theta_T) > 0\,,$$
$$= T_M \text{ if } \text{sgn}\,(p_u \cos\theta_T - p_w \sin\theta_T) < 0\,,$$

i.e. $$T = \tfrac{1}{2}[1-\text{sgn}\,(p_u \cos\theta_T - p_w \sin\theta_T)]\,T_M\,. \tag{68}$$

Also, $H(t)=0$ at all points of an optimal trajectory: this condition gives an equation for t_1.

The adjoint equations, $\dot{p}_u = -\partial H/\partial u$, etc. are:

$$\begin{aligned}\dot{p}_u = (1-p_h)\sin\theta - \frac{\rho S}{2m}\Big[2u(C_L\sin\alpha - C_D\cos\alpha) \\ + V^2\left(\frac{\partial C_L}{\partial u}\sin\alpha - \frac{\partial C_D}{\partial u}\cos\alpha + C_L\frac{\partial(\sin\alpha)}{\partial u} - C_D\frac{\partial(\cos\alpha)}{\partial u}\right)\Big]p_u \\ + rp_v - \Big[q - \frac{\rho S}{2m}\Big\{2u(C_L\cos\alpha + C_D\sin\alpha) \\ + V^2\left(\frac{\partial C_L}{\partial u}\cos\alpha + \frac{\partial C_D}{\partial u}\sin\alpha + C_L\frac{\partial(\cos\alpha)}{\partial u} + C_D\frac{\partial(\sin\alpha)}{\partial u}\right)\Big\}\Big]p_w \\ - \frac{\rho Sl}{2I_y}\left[2uC_m + V^2\frac{\partial C_m}{\partial u}\right]p_q\,,\end{aligned} \tag{69}$$

$$\dot{p}_v = -(1-p_h)\sin\phi\cos\theta - rp_u - \frac{Y_v}{m}p_v + pp_w - \frac{L_v}{I_x}p_p - \frac{N_v}{I_z}p_r\,, \tag{70}$$

$$\begin{aligned}\dot{p}_w = -(1-p_h)\cos\phi\cos\theta - \Big[-q + \frac{\rho S}{2m}\Big\{2w(C_L\sin\alpha - C_D\cos\alpha) \\ + V^2\left(\frac{\partial C_L}{\partial w}\sin\alpha - \frac{\partial C_D}{\partial w}\cos\alpha + C_L\frac{\partial(\sin\alpha)}{\partial w} - C_D\frac{\partial(\cos\alpha)}{\partial w}\right)\Big\}\Big]p_u \\ - pp_v + \frac{\rho S}{2m}\Big[2w(C_L\cos\alpha + C_D\sin\alpha) \\ + V^2\left(\frac{\partial C_L}{\partial w}\cos\alpha + \frac{\partial C_D}{\partial w}\sin\alpha + C_L\frac{\partial(\cos\alpha)}{\partial w} + C_D\frac{\partial(\sin\alpha)}{\partial w}\right)\Big]p_w \\ - \frac{\rho Sl}{2I_y}\left[2wC_m + V^2\frac{\partial C_m}{\partial w}\right]p_q\,,\end{aligned} \tag{71}$$

$$\dot{p}_p = -wp_v + vp_w - \frac{L_p}{I_x}p_p + b_y rp_q + b_z qp_r - p_\phi\,, \tag{72}$$

$$\dot{p}_q = wp_u - up_w + b_x rp_p - \frac{M_q}{I_y}p_q + b_z pp_r - p_\theta\sin\phi\tan\theta - p_\phi\cos\phi\,, \tag{73}$$

$$\dot{p}_r = -vp_u + up_v + b_x qp_p + b_y pp_q - \frac{N_r}{I_z}p_r - p_\phi\cos\phi\tan\theta + p_\phi\sin\phi\,, \tag{74}$$

$$\dot{p}_\phi = -(v\cos\phi - w\sin\phi)\cos\theta\,(1-p_h) - gp_v\cos\phi\cos\theta + gp_w\sin\phi\cos\theta - (q\cos\phi - r\sin\phi)\tan\theta p_\phi + (q\sin\phi + r\cos\phi)p_\theta\,, \tag{75}$$

$$\dot{p}_\theta = (u\cos\theta + v\sin\phi\sin\theta + w\cos\phi\sin\theta)(1-p_h) + gp_u\cos\theta + gp_v\sin\phi\sin\theta + gp_w\cos\phi\sin\theta - (q\sin\phi + r\cos\phi)\sec^2\theta p_\phi\,, \tag{76}$$

$$\begin{aligned}\dot{p}_h &= -\frac{\partial H}{\partial\rho}\frac{d\rho}{dh}\\ &= \frac{SV^2}{2}\left[\frac{1}{m}\{(C_L\sin\alpha - C_D\cos\alpha)p_u - (C_L\cos\alpha + C_D\sin\alpha)p_w\} + \frac{lC_m}{I_y}p_q\right]\frac{\delta\left(\frac{1}{\delta R}-1\right)\rho}{T_0-\delta h}\,.\end{aligned} \tag{77}$$

From equations (17) and (29), we find

$$V^2\frac{\partial(\sin\alpha)}{\partial u} = V^2\frac{\partial(\cos\alpha)}{\partial w} = -u\sin\alpha = -w\cos\alpha\,, \tag{78}$$

$$V^2\frac{\partial(\cos\alpha)}{\partial u} = w\sin\alpha\,,\quad V^2\frac{\partial(\sin\alpha)}{\partial w} = u\cos\alpha\,, \tag{79}$$

$$V^2\frac{\partial C_L}{\partial u} = -w\frac{\partial C_L}{\partial\alpha}\,,\quad V^2\frac{\partial C_L}{\partial w} = u\frac{\partial C_L}{\partial\alpha}\,, \tag{80}$$

with similar expressions involving C_D and C_m. By means of equations (78)–(80), derivatives on the right-hand sides of equations (69) and (71) may be expressed in terms of the familiar aerodynamic quantities $\partial C_L/\partial\alpha$, $\partial C_m/\partial\alpha$, etc. Equations (69) and (71) become, respectively,

$$\begin{aligned}\dot{p}_u = (1-p_h)\sin\theta &- \frac{\rho S}{2m}\left[2u(C_L\sin\alpha - C_D\cos\alpha)\right.\\ &\left.- w\left\{\left(C_L - \frac{\partial C_D}{\partial\alpha}\right)\cos\alpha + \left(C_D + \frac{\partial C_L}{\partial\alpha}\right)\sin\alpha\right\}\right]p_u\\ &+ rp_v - \left[q - \frac{\rho S}{2m}\left\{2u(C_L\cos\alpha + C_D\sin\alpha)\right.\right.\\ &\left.\left.- w\left(\left(C_D + \frac{\partial C_L}{\partial\alpha}\right)\cos\alpha - \left(C_L - \frac{\partial C_D}{\partial\alpha}\right)\sin\alpha\right)\right\}\right]p_w\\ &- \frac{\rho Sl}{2I_y}\left[2uC_m - w\frac{\partial C_m}{\partial\alpha}\right]p_q\,,\end{aligned} \tag{81}$$

$$\begin{aligned}\dot{p}_w = -(1-p_h)\cos\phi\cos\theta &- \left[-q + \frac{\rho S}{2m}\left\{2w(C_L\sin\alpha - C_D\cos\alpha)\right.\right.\\ &\left.\left.+ u\left(\left(C_L - \frac{\partial C_D}{\partial\alpha}\right)\cos\alpha + \left(C_D + \frac{\partial C_L}{\partial\alpha}\right)\sin\alpha\right)\right\}\right]p_u - pp_v\\ &+ \frac{\rho S}{2m}\left[2w(C_L\cos\alpha + C_D\sin\alpha) + u\left\{\left(C_D + \frac{\partial C_L}{\partial\alpha}\right)\cos\alpha - \left(C_L - \frac{\partial C_D}{\partial\alpha}\right)\sin\alpha\right\}\right]p_w\\ &- \frac{\rho Sl}{2I_y}\left[2wC_m + u\frac{\partial C_m}{\partial\alpha}\right]p_q\,.\end{aligned} \tag{82}$$

4. Transversality Conditions

We now consider three specific definitions of spin recovery, and derive the corresponding terminal transversality conditions for the associated optimisation problems. Final values of some of the state variables (or relationships between these final values) have to be specified. Three typical sets of final conditions are:

(a) $u_1 = u_f$, $v_1 = 0$, $w_1 = w_f$, $p_1 = 0$, $q_1 = 0$, $r_1 = 0$, $\phi_1 = 0$,
$\theta_1 = \alpha_f$, i.e. $\tan\theta_1 = w_f/u_f$.

(b) $v_1 = 0$, $p_1 = 0$, $q_1 = 0$, $r_1 = 0$, $\phi_1 = 0$, $u_1 \tan\theta_1 - w_1 = 0$.

(c) $u_1 = u_f$, $v_1 = 0$, $w_1 = w_f$, $p_1^2 + r_1^2 = c$, $\phi_1 = 0$.

The corresponding terminal transversality conditions are given by

$$\boldsymbol{p}_1 = \sum_{i=1}^{9-k} \alpha_i \frac{\partial g_i}{\partial \boldsymbol{x}_1}$$

for arbitrary α_i, where

$$g_i(\boldsymbol{x}_1) = 0, \qquad i = 1, 2, \ldots, 9-k$$

are the terminal conditions on the state variables and $\boldsymbol{p}_1$ is the vector of adjoint variables evaluated at $t = t_1$.

This leads to:

(a) The final values of all the adjoint variables except p_h are arbitrary, and

$$p_{h1} = 0. \tag{83}$$

(b) $p_{v1}, p_{p1}, p_{q1}, p_{r1}, p_{\phi 1}$ are arbitrary.

$$p_{u1} + p_{w1} \tan\theta_1 = 0, \tag{84}$$

$$p_{\theta 1} + p_{w1} u_1 \sec^2\theta_1 = 0, \tag{85}$$

$$p_{h1} = 0 \tag{86}$$

(c) $p_{u1}, p_{v1}, p_{w1}, p_{\phi 1}$ are arbitrary.

$$r_1 p_{p1} - p_1 p_{r1} = 0, \tag{87}$$

$$p_{q1} = p_{\theta 1} = p_{h1} = 0. \tag{88}$$

Note that in each case we have a total of nine terminal conditions on the state and adjoint variables. There are also nine initial conditions on the state variables. The mathematical problem is to solve the eighteen first order differential equations (54)–(62), (70), (72)–(77), (81) and (82) subject to these eighteen initial and terminal conditions, the expressions (65)–(68) for the control variables ξ, η, ζ, T being first substituted into equations (54) and (56)–(59).

The problem just formulated is the classical two-point boundary-value problem of the calculus of variations. Direct methods of solving this problem are available only when the state equations are linear [11]. Thus we are forced to rely on trial and error methods in order to obtain an optimal solution. The method of "backing out of the origin" [12], [13], or more correctly in the present case "backing out of the terminal state", can be used when the number of state variables is not too great. This method is illustrated for a simplified spinning problem in the next Section.

5. Simplified Problem

In order to make the presentation of the method of solution more concise, we now make the additional assumptions that u, T and ρ remain constant throughout the motion, and that V^2 remains constant. (The last condition is consistent with constant u if $w \ll u$).

The state variables are now

$$v, w, p, q, r, \phi, \theta.$$

The state equations are (55)–(61); the control equations are (65), (66), (67); the adjoint equations are (70), (72)–(76) and (82), with $p_u = p_h = 0$ throughout. The transversality conditions are slightly changed since p_u and p_h no longer exist. Referring to Section 4, in case (a) there is now no transversality condition, in case (b) only equation (85) survives, and in case (c) the equation $p_{h1} = 0$ is omitted from equations (88).

It is customary in aerodynamic calculations to write the equations of motion in non-dimensional form. In the present problem, it is convenient to non-dimensionalise both the state and adjoint equations, together with the boundary conditions. The standard British system of non-dimensionalisation known as the "dynamic-normalised form" [6] is used. Thus we obtain the following system of equations:

$$\frac{d\hat{v}}{d\hat{t}} = -\hat{r}\hat{u}+\hat{p}\hat{w}+\hat{g}\sin\phi\cos\theta-\hat{y}_v\hat{v}\,, \tag{89}$$

$$\frac{d\hat{w}}{d\hat{t}} = -\hat{p}\hat{v}+\hat{q}\hat{u}+\hat{g}\cos\phi\cos\theta-(C_L\cos\alpha+C_D\sin\alpha+\hat{T}\cos\theta_T)\,, \tag{90}$$

$$\frac{d\hat{p}}{d\hat{t}} = -b_x\hat{q}\hat{r}-\hat{l}_v\hat{v}-\hat{l}_p\hat{p}-\hat{l}_\xi\,\mathrm{sgn}\,(p_p)\xi_M\,, \tag{91}$$

$$\frac{d\hat{q}}{d\hat{t}} = -b_y\hat{r}\hat{p}+\frac{\mu C_m}{i_y}-\hat{m}_q\hat{q}-\frac{\hat{m}_\eta}{2}\left[(\eta_M+\eta_m)+\mathrm{sgn}\,(p_q)(\eta_M-\eta_m)\right], \tag{92}$$

$$\frac{d\hat{r}}{d\hat{t}} = -b_z\hat{p}\hat{q}-\hat{n}_v\hat{v}-\hat{n}_r\hat{r}-\hat{n}_\zeta\,\mathrm{sgn}\,(p_r)\zeta_M\,, \tag{93}$$

$$\frac{d\phi}{d\hat{t}} = \hat{p}+\hat{q}\sin\phi\tan\theta+\hat{r}\cos\phi\tan\theta\,, \tag{94}$$

$$\frac{d\theta}{d\hat{t}} = \hat{q}\cos\phi-\hat{r}\sin\phi\,, \tag{95}$$

$$\frac{d\hat{p}_v}{d\hat{t}} = -\sin\phi\cos\theta+\hat{y}_v\hat{p}_v+\hat{p}\hat{p}_w+\hat{l}_v\hat{p}_p+\hat{n}_v\hat{p}_r\,, \tag{96}$$

$$\begin{aligned}\frac{d\hat{p}_w}{d\hat{t}} = {} & -\cos\phi\cos\theta-\hat{p}\hat{p}_v+\Bigg[2\hat{w}(C_L\cos\alpha+C_D\sin\alpha) \\ & +\hat{u}\left\{\left(C_D+\frac{\partial C_L}{\partial\alpha}\right)\cos\alpha-\left(C_L-\frac{\partial C_D}{\partial\alpha}\right)\sin\alpha\right\}\Bigg]\hat{p}_w \\ & -\frac{1}{i_y}\left[2\hat{w}C_m+\hat{u}\frac{\partial C_m}{\partial\alpha}\right]\hat{p}_q\,,\end{aligned} \tag{97}$$

$$\frac{d\hat{p}_p}{d\hat{t}} = -\hat{w}\hat{p}_v+\hat{v}\hat{p}_w+\hat{l}_p\hat{p}_p+b_y\hat{r}\hat{p}_q+b_z\hat{q}\hat{p}_r-\hat{p}_\phi\,, \tag{98}$$

$$\frac{d\hat{p}_q}{d\hat{t}} = -u\hat{p}_w+b_x\hat{r}\hat{p}_p+\hat{m}_q\hat{p}_q+b_z\hat{p}\hat{p}_r-\hat{p}_\phi\sin\phi\tan\theta-\hat{p}_\theta\cos\phi\,, \tag{99}$$

$$\frac{d\hat{p}_r}{d\hat{t}} = u\hat{p}_v+b_x\hat{q}\hat{p}_p+b_y\hat{p}\hat{p}_q+\hat{n}_r\hat{p}_r-\hat{p}_\phi\cos\phi\tan\theta+\hat{p}_\theta\sin\phi\,, \tag{100}$$

$$\begin{aligned}\frac{d\hat{p}_\phi}{d\hat{t}} = {} & -(\hat{v}+\hat{g}\hat{p}_v)\cos\phi\cos\theta+(\hat{w}+\hat{g}\hat{p}_w)\sin\phi\cos\theta \\ & -(\hat{q}\cos\phi-\hat{r}\sin\phi)\hat{p}_\phi\tan\theta+(\hat{q}\sin\phi+\hat{r}\cos\phi)\hat{p}_\theta\,,\end{aligned} \tag{101}$$

$$\begin{aligned}\frac{d\hat{p}_\theta}{d\hat{t}} = {} & \hat{u}\cos\theta+(\hat{v}+\hat{g}\hat{p}_v)\sin\phi\sin\theta+(\hat{w}+\hat{g}\hat{p}_w)\cos\phi\sin\theta \\ & -(\hat{q}\sin\phi+\hat{r}\cos\phi)\hat{p}_\phi\sec^2\theta\,.\end{aligned} \tag{102}$$

The initial and final conditions for these equations, together with the transversality conditions, are also put in non-dimensional form.

6. Method of Solution

Consider equations (89)–(102) with given initial conditions on the non-dimensional state variables:

$$\hat{v}=\hat{v}_e,\ \hat{w}=\hat{w}_e,\ \ldots,\ \phi=\phi_e\,,$$

together with final conditions and terminal transversality conditions on these variables. The initial conditions represent a steady spinning motion and hence satisfy equations (89)–(95), with equations (91), (92) and (93) modified so that appropriate control angles (possibly zero) for the steady spin replace $\pm\xi_M$, η_M, η_m or $\pm\zeta_M$. Equations (89)–(102) cannot be solved directly, because we have no information concerning the initial conditions on the adjoint variables.

The method of "backing out of the terminal state" is to reverse the time variable in equations (89)–(102) by writing

$$\hat{t}_R=\hat{t}_1-\hat{t}\,. \tag{103}$$

The new time variable $\hat{t}_R$ is the "time to go". Since $\hat{t}$ does not appear explicitly on the right-hand sides of the equations, the effect of equation (103) is simply to change $d/d\hat{t}$ to $d/d\hat{t}_R$ on the left-hand side and reverse the sign of every term on the right-hand side of each equation. For example, equation (89) becomes

$$\frac{d\hat{v}}{d\hat{t}_R}=\hat{r}\hat{u}-\hat{p}\hat{w}-\hat{g}\sin\phi\cos\theta+\hat{y}_v\hat{v}\,. \tag{89R}$$

The equations with reversed time variable will be described as equations (89R)–(102R). The initial conditions on the state variables for these equations will be the same as the final and terminal transversality conditions on the state variables for equations (89)–(102): these are known. It remains to determine the correct set of initial conditions on the adjoint variables for equations (89R)–(102R).

Note first that any solution of equations (89R)–(102R) satisfying the known initial conditions on the state variables will be an optimal solution for the original problem, since it satisfies the Pontryagin minimum principle, including the terminal transversality conditions, for this problem, i.e. for equations (89)–(102). However, in general, such a solution of equations (89R)–(102R) will not satisfy the correct final conditions on the state variables, these, of course, being the given initial conditions on the state variables for equations (89)–(102).

To overcome this difficulty, a search is made in the space of the adjoint variables: different sets of initial conditions on the adjoint variables for equations (89R)–(102R) are used until the correct (or nearly correct) final conditions are obtained. The required optimal trajectory is then the reverse of the trajectory obtained in this way. The method is described in more detail, with a numerical illustration, in reference [13].

The time required for a systematic search may be prohibitive; on the other hand it should be noted that the only interaction between the state equations (89)–(95) and the adjoint equations (96)–(102) is through the control terms in equations (91)–(93), and these terms depend only on the signs of p_p, p_q and p_r. Computation of a few solutions may therefore indicate that the initial values chosen for many adjoint variables are not critical, so that the search can be confined to a small subset of the adjoint variables.

7. Conclusions

The problem of recovering from a steady spin with minimum loss of height may be formulated as a problem in optimal control theory, to which the Pontryagin minimum principle can be applied. The spin may be represented mathematically in varying degrees of complexity, but though it is always possible to formulate the optimisation problem, its solution demands a trial and error process which can only be accomplished with an acceptable amount of computation for a small number of state variables. Limited computational experience with the method suggests that six or seven state variables are the most that can be used if the search for an optimal trajectory is not to be unduly lengthy.

Apart from the complexity, or otherwise, of the mathematical model, a wide choice of initial and final conditions may be postulated. Also, there is no inherent difficulty in representing the aerodynamic terms with any desired accuracy.

Notation

In general, the aerodynamic notation is based on reference [6], and the optimisation notation on reference [10].

b_x, b_y, b_z	inertia ratios $-(I_y-I_z)/I_x$, $-(I_z-I_x)/I_y$, $-(I_x-I_y)/I_z$
C_L, C_D, C_m	lift, drag and pitching moment coefficients $L/\frac{1}{2}\rho SV^2$, $D/\frac{1}{2}\rho SV^2$, $M/\frac{1}{2}\rho SlV^2$
c_{zx}, c_{yx}, c_{xy} c_{zy}, c_{yz}, c_{xz}	engine momentum parameters J_z/I_x, etc.
D	drag
d_x, d_y, d_z	inertia ratios $-I_{yz}/I_x$, $-I_{yz}/I_y$, $-I_{yz}/I_z$
e_x, e_y, e_z	inertia ratios $-I_{zx}/I_x$, $-I_{zx}/I_y$, $-I_{zx}/I_z$
f_x, f_y, f_z	inertia ratios $-I_{xy}/I_x$, $-I_{xy}/I_y$, $-I_{xy}/I_z$
g	acceleration due to gravity
$\hat{g}$	$mg/\frac{1}{2}\rho_e SV_e^2$
g_x, g_y, g_z	components of g
h	altitude
h_L	height lost
H	Hamiltonian
I_x, I_y, I_z	moments of inertia
J_x, J_y, J_z	components of angular momentum of engine rotors
l	characteristic length of aircraft
L	lift
L, M, N	rolling, pitching and yawing moments
m	mass of aircraft
p, q, r	components of angular velocity
p_u, p_v, etc.	adjoint variables
R	gas constant
S	wing area
t	time (sec.)
$\hat{t}$	t/τ
t_R	time to go
$\hat{t}_R$	t_R/τ
T	thrust
T_0	absolute temperature at sea level
u, v, w	components of velocity
V	$(u^2+w^2)^{\frac{1}{2}}$
X, Y, Z	components of force
α	wing incidence
δ	temperature gradient in atmosphere
θ_T	angle between x-axis and thrust line
μ	relative density parameter $m/\frac{1}{2}\rho_e Sl$
ξ, η, ζ	aileron, elevator and rudder angles
ρ	air density
τ	unit of aerodynamic time $m/\frac{1}{2}\rho_e SV_e$
ϕ, θ, ψ	bank, attitude and azimuth angles

Subscripts, etc.

v_e	datum value of v
v'	increment in v measured from v_e
v_1	final value of v, i.e. $v(t_1)$
$\hat{v}$	(read "v cap"): dynamic-normalised value of v
$\dot{v}$	dv/dt
Y_v, L_p, etc.	aerodynamic derivatives $\partial Y/\partial v$, $\partial L/\partial p$, etc.

Non-dimensionalisation

Non-dimensional quantities are expressed in dynamic-normalised units, and are obtained as follows.

Define units of mass, length and time as m, μl and τ respectively, and divide each dimensional quantity by an appropriate factor depending on its dimensions. In particular, quantities representing force and speed are changed to the dynamic-normalised system by dividing by $\frac{1}{2}\rho_e S V_e^2$ and V_e, respectively.

The same procedure is applied to aerodynamic stability derivatives, with the additional requirements that

(i) their signs are changed,

(ii) an inertia parameter is included in the concise form (as used in this paper) of a dynamic-normalised moment derivative.

Thus the concise forms of the aerodynamic stability derivatives Y_v and L_p in dynamic-normalised units are

$$\hat{y}_v = \frac{-Y_v}{\frac{1}{2}\rho_e S V_e^2/V_e} = \frac{-Y_v}{\frac{1}{2}\rho_e S V_e},$$

$$\hat{l}_p = \frac{-L_p}{m(\mu l)^2/\tau} \Bigg/ \frac{I_x}{m(\mu l)^2} = \frac{-L_p}{\frac{1}{2}\rho_e S l^2 V_e i_x},$$

The dimensions of the adjoint variables may be deduced from the fact that the Hamiltonian, equation (64), has the dimensions of a velocity.

REFERENCES

[1] W. J. Duncan, *The Principles of the Control and Stability of Aircraft.* Chapter 11, Stalling and the Spin, by A. D. Young. Cambridge University Press, 1952.

[2] T. H. Kerr, The spinning of modern aircraft, *Aircraft Engineering,* Vol. XXVII, No. 11, Nov. 1954.

[3] Neihouse, Klinar and Scher, *Status of spin research for recent airplane designs,* N.A.C.A. R.M. L57F12, (T.I.L. 5897), 1957.

[4] S. H. Scher, *et al., Analytical investigation of effect of spin entry technique on spin and recovery characteristics for a 60° delta-wing airplane,* N.A.S.A. T.N. D-156, 1959.

[5] G. R. Walsh, Approximate theories of the spinning of aircraft, *Journ. Eng. Math.* 1, pp. 237–249, 1967.

[6] Royal Aeronautical Society, London, Engineering Sciences Data, Aeronautical Series, Dynamics Sub-Series. 67001 Introduction to notation. Jan. 1967; 67002 Notation. Oct. 1966; 67003 The equations of motion of a rigid aircraft. Sept. 1966;

[7] D. W. Rhoads and J. M. Schuler, A theoretical and experimental study of airplane dynamics in large-disturbance maneuvers, *Journ. Aero. Sci.* 24, pp. 507–526, 532, 1957.

[8] C. D. Perkins and R. E. Hage, *Airplane Performance, Stability and Control,* Wiley, 1949.

[9] G. Leitmann, *An Introduction to Optimal Control,* McGraw-Hill, 1966.

[10] M. Athans and P. L. Falb, *Optimal Control,* McGraw-Hill, 1966.

[11] B. Paiewonsky, *Synthesis of Optimal Controls.* Chapter 9 in *Topics in Optimization* (G. Leitmann, ed.), Academic Press, 1967.

[12] E. B. Lee and L. Markus, *Foundations of Optimal Control Theory,* Wiley, 1967.

[13] G. R. Walsh, Time-optimal rotational motion, *Journ. Eng. Math.* 3, pp. 95–101, April, 1969.